arXiv:hep-th/0402020v1 3 Feb 2004

HU Berlin-EP-04/08

Comments on the scalar propagator in AdS $\times$ S and the BMN plane wave

H. Dorn, M. Salizzoni, C. Sieg[1]

[1] Humboldt–Universität zu Berlin, Institut für Physik
Newtonstraße 15, D-12489 Berlin

Abstract: We discuss the scalar propagator on generic $\mathrm{AdS}_{d+1} \times \mathrm{S}^{d'+1}$ backgrounds. For the conformally flat situations and masses corresponding to Weyl invariant actions the propagator is powerlike in the sum of the chordal distances with respect to AdS_{d+1} and $\mathrm{S}^{d'+1}$. In all other cases the propagator depends on both chordal distances separately. We discuss the KK mode summation to construct the propagator in brief. For $\mathrm{AdS}_5 \times \mathrm{S}^5$ we relate our propagator to the expression in the BMN plane wave limit and find a geometric interpretation of the variables occurring in the known explicit construction on the plane wave.

1 Introduction

The AdS/CFT correspondence relates $\mathcal{N} = 4$ super Yang-Mills gauge theory in Minkowski space to type II B string theory in $\mathrm{AdS}_5 \times \mathrm{S}^5$ with some RR background flux. Explicit tests of the AdS/CFT correspondence, beyond the supergravity approximation, remain a difficult task, since the relevant string spectrum in general is not available. In a limit proposed by Berenstein, Maldacena and Nastase (BMN limit) [1] the $\mathrm{AdS}_5 \times \mathrm{S}^5$ background itself is transformed via a Penrose limit to a certain plane wave spacetime [2–4]. In this background string theory is exactly quantizable, and thus enables independent checks of the duality, including string effects. In this BMN plane wave spacetime the separation between the AdS_5 and the S^5 part breaks down, and one has to take the limit on full 10-dimensional $\mathrm{AdS}_5 \times \mathrm{S}^5$ objects.

One of the crucial unsolved questions in this setting concerns the issue of holography [5–9] . In the Penrose limiting process the old 4-dimensional conformal boundary is put beyond the new plane wave space, which by itself has a one-dimensional conformal boundary. In [10] we started to investigate this issue and found for each point remaining in the final plane wave a degeneration of the cone of boundary reaching null geodesics into a single direction. To continue this program beyond geometric properties, we now want to study the limiting process for field theoretical propagators. The scalar propagator in the BMN plane wave has been constructed in [11] by a direct approach leaving the issue of its derivation via a limiting process from $\mathrm{AdS}_5 \times \mathrm{S}^5$ as an open problem.

Here we will discuss the construction of the scalar propagator on $\mathrm{AdS}_{d+1}\times \mathrm{S}^{d'+1}$ spaces with radii R_1 and R_2, respectively. Allowing for generic dimensions d and d' as well as generic curvature radii R_1 and R_2 is very helpful to understand the general mechanism for the construction of the propagator.

In section 2 we discuss the differential equation for the propagator and its solution in conformally flat backgrounds and for masses corresponding to Weyl invariant actions. We will compare with the pure AdS_{d+1} case and interpret the results from a flat space perspective.

In section 3 we present the KK mode summation to construct the propagator and describe how this leads to a theorem summing up Legendre and Gegenbauer functions.

In section 4 we will discuss the plane wave limit of $\mathrm{AdS}_5\times \mathrm{S}^5$ in brief. We will explicitly show that the massless propagator on the full space indeed reduces to the expression of [11].

2 The differential equation for the propagator and its solution

The scalar propagator is defined as the solution of

$$(\Box_z - M^2)G(z,z') = \frac{1}{\sqrt{-g}}\delta(z,z') \ , \tag{1}$$

with suitable boundary conditions at infinity. $\Box_z$ denotes the d'Alembert operator on $\mathrm{AdS}_{d+1}\times \mathrm{S}^{d'+1}$ acting on the first argument of the propagator $G(z,z')$. In the following we denote the coordinates referring to the AdS_{d+1} factor by x and those referring to the $\mathrm{S}^{d'+1}$ factor by y, i.e. $z=(x,y)$. AdS_{d+1} and $\mathrm{S}^{d'+1}$ can be interpreted as embeddings respectively in $\mathbb{R}^{2,d}$ and in $\mathbb{R}^{d'+2}$ with the help of the constraints[1]

$$-X_0^2 - X_{d+1}^2 + \sum_{i=1}^{d} X_i^2 = -R_1^2 \ , \qquad \sum_{i=1}^{d'+2} Y_i^2 = R_2^2 \ , \tag{2}$$

where $X=X(x)$, $Y=Y(y)$ depend on the coordinates x and y, respectively. We define the chordal distances on both spaces to be

$$u(x,x') = (X(x)-X(x'))^2 \ , \qquad v(y,y') = (Y(y)-Y(y'))^2 \ . \tag{3}$$

The distances have to be computed with the corresponding flat metrics of the embedding spaces that can be read off from (2). The chordal distance u is a unique function of x and x' if one restricts oneself to the hyperboloid. On the universal covering it is continued as a periodic function.

Using the homogeneity and isotropy of both AdS_{d+1} and $\mathrm{S}^{d'+1}$ it is clear that the propagator can depend on z, z' only via the chordal distances $u(x,x')$ and $v(y,y')$.[2]. The d'Alembert operator is a direct sum of the parts on both subspaces $\Box_z = \Box_x + \Box_y$. $\Box_x$ and $\Box_y$ can be expressed with the help of u and v and first and second derivatives w.r.t u and v respectively. One can now ask for a solution of (1) that only depends on the total chordal distance $u+v$. It is easy to derive that such a solution exists if and only if

$$R_1 = R_2 = R \ , \qquad M^2 = \frac{d'^2 - d^2}{4R^2} \ . \tag{4}$$

[1]More precisely, AdS_{d+1} is the universal covering of the hyperboloid in $\mathbb{R}^{2,d}$.
[2]Strictly speaking this at first applies only if AdS_{d+1} is restricted to the hyperboloid

One first looks for a solution of the homogeneous version of (1) and then checks its short distance singularity such that it generates the δ-function on the r.h.s. of (1). Hence after fixing the normalization we end up with

$$G(z,z') = \frac{\Gamma(\frac{d+d'}{2})}{4\pi^{\frac{d+d'}{2}+1}} \frac{1}{(u+v+i\epsilon)^{\frac{d+d'}{2}}} \ . \tag{5}$$

Under the conditions (4) there is a second solution of (1), but with the δ-source shifted to another position, that depends only on $(u-v)$.

$$\tilde{G}(z,z') \propto \frac{1}{(u-v+4R^2)^{\frac{d+d'}{2}}} \ . \tag{6}$$

Both above given results have the same asymptotic falloff. The exponent in the denominator is just equal to $\Delta_+(d,M^2)$. Where

$$\Delta_\pm(d,m^2) = \frac{1}{2}\Big(d \pm \sqrt{d^2+4m^2R_1^2}\Big) \tag{7}$$

are the conformal dimensions of CFT fields related to the scalar fields via the AdS/CFT correspondence.

The above results can be compared with the propagators in pure AdS_{d+1}. One finds simple powers of the chordal distance u for the mass value

$$m^2 = \frac{1-d^2}{4R_1^2} \ . \tag{8}$$

The two solutions are

$$\begin{aligned} \frac{1}{2}(G_{\Delta_-}+G_{\Delta_+}) &= \frac{\Gamma(\frac{d-1}{2})}{4\pi^{\frac{d+1}{2}}} \frac{1}{u^{\frac{d-1}{2}}} \ , \\ \frac{1}{2}(G_{\Delta_-}-G_{\Delta_+}) &= \frac{\Gamma(\frac{d-1}{2})}{4\pi^{\frac{d+1}{2}}} \frac{1}{(u+4R^2)^{\frac{d-1}{2}}} \ . \end{aligned} \tag{9}$$

Where the linear combinations on the l.h.s. refer to the expression of the general massive scalar propagator on pure AdS_{d+1} space [12, 13]

$$G_{\Delta_\pm}(x,x') = \frac{\Gamma(\Delta_\pm)}{R_1^{d-1}2\pi^{\frac{d}{2}}\Gamma(\Delta_\pm-\frac{d}{2}+1)}\Big(\frac{\xi}{2}\Big)^{\Delta_\pm} F\big(\tfrac{\Delta_\pm}{2},\tfrac{\Delta_\pm}{2}+\tfrac{1}{2};\Delta_\pm-\tfrac{d}{2}+1;\xi^2\big)\ , \quad \xi = \frac{2R_1^2}{u+2R_1^2}\ , \tag{10}$$

evaluated for $\Delta_\pm = \frac{d\pm1}{2}$ that corresponds to the mass value (8). The powerlike solution with the correct short distance behavior is given by the first line in (10). The second combination resembles (6). In contrast to the $\mathrm{AdS}_{d+1}\times\mathrm{S}^{d'+1}$ case here the exponent of u is given by Δ_-.

At the end of this section we give a simple interpretation of the conditions (4). The equality of the radii is exactly the condition for conformal flatness of the complete product space $\mathrm{AdS}_{d+1}\times\mathrm{S}^{d'+1}$ as a whole. The mass values in (4) and (8) are generated by the Weyl invariant coupling of the scalar field to the background. Therefore one can use a Weyl transformation to map (5) and (6) and (9) to flat space solutions. The Poincaré patch of $\mathrm{AdS}_{d+1}\times\mathrm{S}^{d'+1}$ is mapped to $\mathbb{R}^{1,d+d'+1}$ with the boundary of $\mathrm{AdS}_{d+1}\times\mathrm{S}^{d'+1}$ mapped to a

certain d-dimensional subspace $\mathbb{R}^{1,d-1}$. The Poincaré patch of pure AdS_{d+1} is mapped to a flat half space $\mathbb{R}^{1,d}_{+}$ with the boundary of AdS_{d+1} mapped to the boundary of the half space. Therefore in the pure AdS_{d+1} case we can use the standard mirror charge method to implement either Dirichlet or Neumann boundary conditions. These two solutions refer to the two values $\Delta_\pm$. In the $\mathrm{AdS}_{d+1}\times \mathrm{S}^{d'+1}$ case the mirror point lies inside the patch and therefore the mirror charge method is not applicable and the expression (5) is the single solution. This observation fits nicely with the fact that in AdS spaces one has to respect the Breitenlohner-Freedman bounds [14,15]. In the above conformally flat and Weyl invariant coupled cases the bounds teach us that two solutions (with Δ_+ and Δ_-) are allowed in the pure AdS_{d+1} space whereas only one solution (scaling with Δ_+) is allowed in $\mathrm{AdS}_{d+1}\times \mathrm{S}^{d'+1}$.

3 Mode summation on $\mathbf{AdS}_{d+1}\times \mathbf{S}^{d'+1}$

In this section we will use the propagator on pure AdS_{d+1} (10) and the spherical harmonics on $\mathrm{S}^{d'+1}$ to construct the propagator on $\mathrm{AdS}_{d+1}\times \mathrm{S}^{d'+1}$ via its mode expansion, summing up all the KK modes. The mode summation allows for a relaxation of the conditions (4): a conformally flat background ($R_1 = R_2$) is no longer required but the restriction to a special mass value remains to ensure that the conformal dimensions $\Delta_\pm$ are linear functions of l [3], with l denoting the lth mode in the KK tower. This is the condition to find an explicit expression for the sum and it is fulfilled in many cases.[4] In the literature it is believed that an explicit computation of the KK mode summation is too cumbersome [11]. Here we will compare the mode summation in the $\mathrm{AdS}_{d+1}\times \mathrm{S}^{d'+1}$ case for equal radii with the solution of the differential equation (5). This leads to a theorem for summing certain products of Legendre and Gegenbauer functions. An explicit discussion of the $\mathrm{AdS}_3\times \mathrm{S}^3$ case with generic radii and an explanation of how to deal with the mode summation can be found in [19].

For the solution of (1) we make the following ansatz

$$G(z,z') = \frac{1}{R_2^{d'+1}} \sum_I G_I(x,x') Y^I(y) Y^I(y') \ , \tag{11}$$

where we sum over the multiindex $I = (l, m_1, \ldots, m_{d'})$ such that $l \geq m_1 \geq \cdots \geq m_{d'-1} \geq |m_{d'}| \geq 0$ and Y^I denote the spherical harmonics on $\mathrm{S}^{d'+1}$ that are eigenfunctions with respect to the Laplacian on the sphere

$$\Box_y Y^I(y) = -\frac{l(l+d')}{R_2^2} Y^I(y) \ . \tag{12}$$

The mode dependent Green's function on AdS_{d+1} then fulfills

$$\Big(\Box_x - m^2\Big) G_I(x,x') = \frac{1}{\sqrt{-g_{\mathrm{AdS}}}} \delta(x,x') \ , \qquad m^2 = M^2 + m^2_{\mathrm{KK}} = M^2 + \frac{l(l+d')}{R_2^2} \ . \tag{13}$$

The solution of this equation was already given in (10), into which the (now KK mode dependent) conformal dimensions defined in (7) enter. In the conformally flat case at the

[3]This condition is necessary to evaluate the KK mode sum explicitly.

[4]E. g. in type II B supergravity on $\mathrm{AdS}_5\times \mathrm{S}^5$ the $\Delta_\pm$ of the scalar modes corresponding to the CPOs and descendant operators depend linearly on l [16–18].

Weyl invariant mass value (4) the conformal dimensions are given by $\Delta = \Delta_+ = l + \frac{d+d'}{2}$. The propagator is expressed as

$$
\begin{aligned}
G(z,z') = {} & \frac{\Gamma(\frac{d'}{2})}{4\pi}\Big(\frac{\xi}{2\pi R^2}\Big)^{\frac{d+d'}{2}} \\
& \times \sum_{l=0}^{\infty} \frac{\Gamma(l+\frac{d+d'}{2})}{\Gamma(l+\frac{d'}{2})}\Big(\frac{\xi}{2}\Big)^l F\big(\tfrac{l}{2}+\tfrac{d+d'}{4}, \tfrac{l}{2}+\tfrac{d+d'}{4}+\tfrac{1}{2}; l+\tfrac{d'}{2}+1; \xi^2\big) C_l^{(\frac{d'}{2})}(1-\tfrac{v}{2R^2}) \ ,
\end{aligned}
\tag{14}
$$

where (10) and the completeness relation for the spherical harmonics has been used. The evaluation of the above sum must precisely lead to the simple result (5). In [19] we have proven a theorem that allows to perform the above summation with integer d and d' as a special case.

4 The plane wave limit

The BMN plane wave background arises as a certain Penrose limit of $\mathrm{AdS}_5 \times \mathrm{S}^5$. The scalar propagator in the plane wave has been constructed in [11]. In this section we study how this propagator in the massless case arises as a limit of our $\mathrm{AdS}_5 \times \mathrm{S}^5$ propagator (5).

This approach is in the spirit of [10], where one follows the limiting process instead of taking the limit before starting any computations. One finds a simple interpretation of certain functions of the coordinates introduced in [11].

Taking the aforementioned Penrose limit of $\mathrm{AdS}_5 \times \mathrm{S}^5$ means to focus into the neighbourhood of a certain null geodesic which runs along an equator of the sphere with velocity of light. The metric of $\mathrm{AdS}_5 \times \mathrm{S}^5$ in global coordinates

$$
ds^2 = R^2\big(-dt^2\cosh^2\rho + d\rho^2 + \sinh^2\rho d\Omega_3^2 + d\psi^2\cos^2\vartheta + d\vartheta^2 + \sin^2\vartheta d\tilde{\Omega}_3^2\big) \tag{15}
$$

via the replacements

$$
t = z^+ + \frac{z^-}{R^2}\ , \qquad \psi = z^+ - \frac{z^-}{R^2}\ , \qquad \rho = \frac{r}{R}\ , \qquad \vartheta = \frac{y}{R} \tag{16}
$$

in the $R \to \infty$ limit turns into the BMN plane wave metric.

After expressing the chordal distances (3) in global coordinates and applying (16) one gets at large R up to terms vanishing for $R \to \infty$

$$
\begin{aligned}
u &= 2R^2\Big[-1 + \cos\Delta z^+ + \frac{1}{R^2}\Big(-(\vec{x}^2+\vec{x}'^2)\sin^2\frac{\Delta z^+}{2} + \frac{(\vec{x}-\vec{x}')^2}{2} - \Delta z^-\sin\Delta z^+\Big)\Big] \\
v &= 2R^2\Big[+1 - \cos\Delta z^+ + \frac{1}{R^2}\Big(-(\vec{y}^2+\vec{y}'^2)\sin^2\frac{\Delta z^+}{2} + \frac{(\vec{y}-\vec{y}')^2}{2} - \Delta z^-\sin\Delta z^+\Big)\Big]\ ,
\end{aligned}
\tag{17}
$$

where $\Delta z^\pm = z^\pm - z'^\pm$. In the $R \to \infty$ limit the sum of both chordal distances is thus given by

$$
\Phi = \lim_{R\to\infty}(u+v) = -2(\vec{z}^2+\vec{z}'^2)\sin^2\frac{\Delta z^+}{2} + (\vec{z}-\vec{z}')^2 - 4\Delta z^-\sin\Delta z^+\ , \tag{18}
$$

where $\vec{z} = (\vec{x},\vec{y})$, $\vec{z}' = (\vec{x}',\vec{y}')$ and Φ refers to the notation of [11]. Φ is precisely the $R \to \infty$ limit of the total chordal distance on $\mathrm{AdS}_5 \times \mathrm{S}^5$, which remains finite as both $\sim R^2$ terms in (17) cancel. This happens due to the expansion around a *null* geodesic.

The massless propagator in the plane wave background in the $R \to \infty$ limit of (5) with $d = d' = 4$ thus becomes

$$G_{\mathrm{pw}}(z, z') = \frac{3}{2\pi^5} \frac{1}{(\Phi + i\epsilon)^4} , \tag{19}$$

which agrees with [11].

5 Conclusions

In this paper we have focussed on the propagator of scalar fields on $\mathrm{AdS}_{d+1} \times \mathrm{S}^{d'+1}$ backgrounds. We have discussed the defining wave equation with δ-source in this background. On conformally flat backgrounds for Weyl invariant coupled fields the propagator is simply powerlike in the sum of both chordal distances. An interpretation from the flat space point of view was given using the mirror charge method, resulting in an explanation why on pure AdS one finds two solutions with different asymptotic behavior whereas in $\mathrm{AdS}_{d+1} \times \mathrm{S}^{d'+1}$ there is a unique solution.

In addition for $\mathrm{AdS}_{d+1} \times \mathrm{S}^{d'+1}$ we have investigated the KK decomposition of the propagator using spherical harmonics. In brief we described how a comparison with the solution of the differential equation lead to the formulation of a theorem that sums a product of a Legendre and a Gegenbauer function.

For $\mathrm{AdS}_5 \times \mathrm{S}^5$ backgrounds we explicitly performed the Penrose limit on our expression for the propagator to find the result on the plane wave background in agreement with [11]. The coordinate dependence is given by the $R \to \infty$ limit of the total chordal distance of $\mathrm{AdS}_5 \times \mathrm{S}^5$.

Clearly future work is necessary to construct the propagator for the case of generic mass values. But already with our results one should be able to address the issue of defining a bulk-to-boundary propagator and study its behavior in the plane wave limit.

Acknowledgement The work was supported by DFG (German Science Foundation) with the "Schwerpunktprogramm Stringtheorie" and the "Graduiertenkolleg 271".

References

[1] D. Berenstein, J. M. Maldacena, and H. Nastase, "Strings in flat space and pp waves from $\mathcal{N} = 4$ super Yang Mills," *JHEP* **04** (2002) 013, hep-th/0202021.

[2] M. Blau, J. Figueroa-O'Farrill, C. Hull, and G. Papadopoulos, "A new maximally supersymmetric background of IIB superstring theory," *JHEP* **01** (2002) 047, hep-th/0110242.

[3] M. Blau, J. Figueroa-O'Farrill, C. Hull, and G. Papadopoulos, "Penrose limits and maximal supersymmetry," *Class. Quant. Grav.* **19** (2002) L87–L95, hep-th/0201081.

[4] M. Blau, J. Figueroa-O'Farrill, and G. Papadopoulos, "Penrose limits, supergravity and brane dynamics," *Class. Quant. Grav.* **19** (2002) 4753, hep-th/0202111.

[5] D. Berenstein and H. Nastase, "On lightcone string field theory from super Yang-Mills and holography," hep-th/0205048.

[6] S. R. Das, C. Gomez, and S.-J. Rey, "Penrose limit, spontaneous symmetry breaking and holography in pp-wave background," *Phys. Rev.* **D66** (2002) 046002, `hep-th/0203164`.

[7] R. G. Leigh, K. Okuyama, and M. Rozali, "PP-waves and holography," *Phys. Rev.* **D66** (2002) 046004, `hep-th/0204026`.

[8] E. Kiritsis and B. Pioline, "Strings in homogeneous gravitational waves and null holography," *JHEP* **08** (2002) 048, `hep-th/0204004`.

[9] S. Dobashi, H. Shimada, and T. Yoneya, "Holographic reformulation of string theory on AdS(5) x S(5) background in the PP-wave limit," `hep-th/0209251`.

[10] H. Dorn and C. Sieg, "Conformal boundary and geodesics for AdS(5) x S(5) and the plane wave: Their approach in the Penrose limit," *JHEP* **04** (2003) 030, `hep-th/0302153`.

[11] S. D. Mathur, A. Saxena, and Y. K. Srivastava, "Scalar propagator in the pp-wave geometry obtained from AdS(5) x S(5)," *Nucl. Phys.* **B640** (2002) 367–378, `hep-th/0205136`.

[12] C. P. Burgess and C. A. Lütken, "Propagators and effective potentials in anti-de Sitter space," *Phys. Lett.* **B153** (1985) 137.

[13] E. D'Hoker and D. Z. Freedman, "Supersymmetric gauge theories and the AdS/CFT correspondence," `hep-th/0201253`.

[14] P. Breitenlohner and D. Z. Freedman, "Positive energy in anti-de Sitter backgrounds and gauged extended supergravity," *Phys. Lett.* **B115** (1982) 197.

[15] P. Breitenlohner and D. Z. Freedman, "Stability in gauged extended supergravity," *Ann. Phys.* **144** (1982) 249.

[16] H. J. Kim, L. J. Romans, and P. van Nieuwenhuizen, "The mass spectrum of chiral N=2 d = 10 supergravity on S(5)," *Phys. Rev.* **D32** (1985) 389.

[17] M. Gunaydin and N. Marcus, "The spectrum of the S(5) compactification of the chiral $\mathcal{N} = 2$, $D = 10$ supergravity and the unitarity supermultiplets of U(2, 2/4)," *Class. Quant. Grav.* **2** (1985) L11.

[18] S.-M. Lee, S. Minwalla, M. Rangamani, and N. Seiberg, "Three-point functions of chiral operators in $D = 4$, $\mathcal{N} = 4$ SYM at large N," *Adv. Theor. Math. Phys.* **2** (1998) 697–718, `hep-th/9806074`.

[19] H. Dorn, M. Salizzoni, and C. Sieg, "On the propagator of a scalar field on AdS x S and on the BMN plane wave," `hep-th/0307229`.